Vademecum for

Loading Greens functions

HGS

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This document accompanies program `Green/p/m/greenm.f` and its workhorse subroutines `Green/p/kummer.f` and `Green/p/greens.f`.

1 Disk factor

The expansion coefficients for Farrell's disk are

$$\Gamma_n = \frac{2n+1}{4\pi a^2}\left[-\frac{1+\cos\alpha}{n(n+1)\sin\alpha}\,\frac{\partial P_n(\cos\alpha)}{\partial\alpha}\right] \tag{1}$$

$$= \frac{2n+1}{4\pi a^2} D_n \tag{2}$$

representing a disk of radius α. The quantity in the bracket is the expression running in the infinite sums. The sum over the D's is corresponds (is proportional) to the potential of the disk

$$\sum_{n=0}^{\infty} D_n\, t^n\, P_n(\cos\theta) = \iint_{\mathcal{D}} \frac{\sin\phi}{\sqrt{1-2t\cos\theta'+t^2}}\, d\phi\, d\lambda \tag{3}$$

with θ' the angle from the field point (at θ from the disk's centre) to the infinitesimal volume in the integration.

The derivative of a Legendre polynomial with respect to the angle is conveniently computed from the recurrence relation

$$\frac{\partial P_n(\cos\theta)}{\partial\theta} = \frac{n}{\sin\theta}\left[\cos\theta\, P_n(\cos\theta) - P_{n-1}(\cos\theta)\right] \tag{4}$$

Just for curiosity we could as for the potential of this disk at arbitrary distance. After some algebra,

$$= 2\pi \sum_{n=0}^{\infty} P_n(\cos\theta) \int_0^{\alpha} \frac{\partial}{\partial\phi} \frac{1}{\sqrt{1-2t\cos\phi+t^2}}\, \frac{\partial P_n(\cos\phi)}{\partial\phi}\, \frac{1+\cos\phi}{n(n+1)\sin\phi}\, \sin\phi\, d\phi \tag{5}$$

This is difficult even if $t = 1$. In essence, if we use a disk factor, then also the Kummer-sums should be integrated over the disk, else we have a mixed situation of a finite sum with a disk factor and an infinite one for a point load. It could be solved in cylindrical coordinates, $2a \sin\theta/2 \simeq r$.

The Kummer-transformed Greens function sum with the disk factor included is

$$u(\theta) \simeq \frac{G\,m}{g\,a} \left\{ \left[\sum_{n=0}^{N} (h_n - h_\infty) D_n t^n P_n(\cos\theta) \right] + h_\infty \left[\sum_{n=0}^{\infty} D_n t^n P_n(\cos\theta) \right] \right\} \tag{6}$$

This relation might be useful for computing the Newtonian potential of the disk at a height.

Considering loading effects, we should not combine $t < 1$ with the disk factor.

The disk factor is primarily useful for the ocean tide-generating potential.

However we will present the Kummer transform terms (the analytic expressions) for the case $t < 1$.

2 Summing up Love for the Greens

Just for completeness, here's the recursion formula

$$P_{n+1}(x) = \frac{2n+1}{n+1} x\, P_n(x) - \frac{n}{n+1} P_{n-1}(x) \tag{7}$$

so if you are heading for Clenshaw summation,

$$\alpha_k = x \frac{2n+1}{n+1} \qquad \beta_k = -\frac{n}{n+1} \tag{8}$$

you do the following recursion:

$$y_{n+2} = y_{n+1} = 0 \tag{9}$$

$$y_k = \alpha_k y_{k+1} + \beta_{k+1} y_{k+2} + c_k \qquad \text{for } k = n..1 \text{ step } -1 \tag{10}$$

$$S = y_1 P_1(x) + (\beta_1 y_2 + c_0) P_0(k) = y_1 \cos\theta + c_0 - \frac{1}{2} y_2 \tag{11}$$

In the case of the derivatives

$$\frac{\partial P_{n+1}(\cos\theta)}{\partial\theta} = \frac{2n+1}{n} \cos\theta\, \frac{\partial P_n(\cos\theta)}{\partial\theta} - \frac{n+1}{n} \frac{\partial P_{n-1}(\cos\theta)}{\partial\theta} \tag{12}$$

Remember that in some sums the Love numbers are zero at $n = 0$ and/or $n = 1$ so that you must stop the iteration one step ahead. You need $P_2(x)$ and $P_3(x)$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{13}$$

$$\frac{\partial P_2(\cos\theta)}{\partial\theta} = -3 \sin\theta\, \cos\theta \qquad \frac{\partial P_3(\cos\theta)}{\partial\theta} = \frac{3}{2}(1 - 5\cos^2\theta) \sin\theta \tag{14}$$

3 Kummer-transformed sums

We denote the ocean tide-effective potential with OTEP, and a potential-like entity for horizontal displacement with HDP (a.k.a. PTANG).

The following cases are given, owing to the asymptotic behaviour as $n \to \infty$:

$$h_n \to h_\infty, \quad n\,k_n \to k_\infty, \quad n\,l_n \to l_\infty \tag{15}$$

For vertical displacement, gravity, and OTEP,

$$K_1(t) := \sum_{n=0}^{\infty} t^n P_n(\cos\theta) = \frac{1}{\sqrt{1 - 2t\cos\theta + t^2}} \tag{16}$$

and in the limit $t \to 1$

$$\frac{1}{\sqrt{1 - 2t\cos\theta + t^2}} \to \frac{1}{2\sin\theta/2} \tag{17}$$

The angular derivative of the above is needed for tilt and strain

$$K_{1,\theta}(t) := \frac{\partial}{\partial\theta}\frac{1}{\sqrt{1 - 2t\cos\theta + t^2}} = -\frac{t\sin\theta}{(1 - 2t\cos\theta + t^2)^{3/2}} \tag{18}$$

and in the limit $t \to 1$

$$= -\frac{\sin\theta}{8\sin^3\theta/2} = -\frac{\cot\theta/2}{4\sin\theta/2} \tag{19}$$

For gravity

$$\begin{aligned} k_\infty \sum_{n=1}^{\infty} \frac{n+1}{n} t^n P_n(\cos\theta) &= k_\infty \left[\frac{1}{\sqrt{1 - 2t\cos\theta + t^2}} - 1 + \sum_{n=1}^{\infty} \frac{t^n}{n} P_n(\cos\theta)\right] \\ &= k_\infty \left[K_1(t) - 1 + K_2(t)\right] \end{aligned} \tag{20}$$

since $k_0 = 0$. $K_2(t)$ will evaluated in equation (27).

For OTEP and HDP, $K_2(t)$ applies again,

$$\left\{\begin{array}{c} k_\infty \\ l_\infty \end{array}\right\} \sum_{n=1}^{\infty} \frac{t^n}{n} P_n(\cos\theta) \tag{21}$$

see equation (27).

For tangential displacement we denote

$$S_{2,\theta}(l) := \sum_{n=1}^{\infty} l_n t^n \frac{\partial P_n(\cos\theta)}{\partial\theta} \tag{22}$$

and since $nl_n \to l_\infty$, we need the infinite sum

$$K_{2,\theta}(t) := \frac{\partial}{\partial\theta} \sum_{n=1}^{\infty} \frac{t^n}{n} P_n(\cos\theta) \tag{23}$$

3.1 The 1-over-*n* sum

$$\sum_{n=1}^{\infty} \frac{t^n}{n} P_n = \sum_{n=1}^{\infty} \left[\int_0^t \tau^{n-1} \, d\tau \right] P_n \tag{24}$$

$$= \int_0^t \frac{1}{\tau} \sum_{n=1}^{\infty} \tau^n P_n \, d\tau \tag{25}$$

$$= \int_0^t \frac{1}{\tau} \left[\frac{1}{\sqrt{1 - 2\tau \cos\theta + \tau^2}} - 1 \right] \, d\tau \tag{26}$$

$$= K_{2,\theta}(t) := \log 2 - \log \left[1 - t\cos(\theta) + \sqrt{1 - 2t\cos\theta + t^2} \right] \tag{27}$$

The limit for $t \to 1$ is

$$-\log \left[\sin^2 \theta/2 \; + \; \sin\theta/2 \right] \tag{28}$$

The derivative of (27) with respect to θ is

$$K_{2,\theta}(t) := \frac{\partial}{\partial\theta} \sum_{n=1}^{\infty} \frac{t^n}{n} P_n(\cos\theta) =$$

$$= -\frac{t \, \sin\theta}{1 - t\cos\theta + \sqrt{1 - 2t\cos\theta + t^2}} \left[1 + \frac{1}{\sqrt{1 - 2t\cos\theta + t^2}} \right] \tag{29}$$

In the limit $t \to 1$

$$= -\frac{1}{2} \cot\theta/2 \, \frac{1 + 2\sin\theta/2}{1 + \sin\theta/2} \tag{30}$$

3.2 Strain

Common to the strain components is the sum

$$S_{dd}(l) := \sum_{n=1}^{\infty} n(n+1) l_n t^n P_n(\cos\theta) \tag{31}$$

It occurs in

$$\epsilon_{rr} = -\frac{\nu}{1-\nu}\epsilon_a \tag{32}$$

where ϵ_a is areal strain, and in $\epsilon_{\theta\theta}$ (in our notation NN-STRAIN)

$$S_{\theta\theta}(h,l) = \sum_{n=0}^{\infty} h_n t^n P_n(\cos\theta) + \sum_{n=1}^{\infty} l_n t^n \frac{\partial^2 P_n(\cos\theta)}{\partial\theta^2} =$$

$$= S_1(h) - \cot\theta \sum_{n=1}^{\infty} l_n t^n \frac{\partial P_n(\cos\theta)}{\partial\theta} - \sum_{n=1}^{\infty} n(n+1) l_n t^n P_n(\cos\theta) \tag{33}$$

by means of Legendre's differential equation. The $n(n+1)$-term suggests to evaluate the finite sums together

$$s_n = -(n l_n - l_\infty)\, t^n \left[\frac{\cot\theta}{n}\frac{\partial P_n(\cos\theta)}{\partial\theta} + (n+1) P_n(\cos\theta)\right] \tag{34}$$

Let's call the sum

$$S_{4,\theta}(l) := \sum_{n=1}^{\infty} s_n \tag{35}$$

After Kummer transformation, the sums that must be evaluated are

$$\begin{aligned} K_{4,\theta}(t) &= -\cot\theta \sum_{n=1}^{\infty} \frac{t^n}{n}\frac{\partial P_n(\cos\theta)}{\partial\theta} - \sum_{n=1}^{\infty} (n+1)\, t^n P_n(\cos\theta) && (36) \\ &= \cot\theta\, K_{2,\theta}(t) - K_3(t) \end{aligned}$$

The first term has already been treated. The second term is

$$\begin{aligned} K_3(t) &= \sum_{n=1}^{\infty} \frac{\partial t^{n+1}}{\partial t} P_n(\cos\theta) && (37) \\ &= \frac{\partial}{\partial t}\left(\frac{t}{\sqrt{1-2t\cos\theta+t^2}} - t\right) && (38) \\ &= \frac{1-t\cos\theta}{(1-2t\cos\theta+t^2)^{3/2}} - 1 && (39) \end{aligned}$$

and for $t \to 1$

$$= \frac{1}{4\sin\theta/2} - 1 \tag{40}$$

Combining the analytical terms for (36) and letting $t \to 1$ yields

$$= (1 - \cot\theta/2)\ \cot\theta/2\, \frac{1+2\sin\theta/2}{4+4\sin\theta/2} + 1 - \frac{1}{4\sin\theta/2} \tag{41}$$

The complete Kummer expression for $S_{\theta\theta}$ is

$$K_{\theta\theta} = h_\infty K_1 - l_\infty(\cot\theta\, K_{2,\theta} \; + \; K_3) \tag{42}$$

The last strain component to consider is $\epsilon_{\lambda\lambda}$. See Farrell (1972), equations (50) to (56). There is also shown that the off-diagonal strain elements are zero.

$$\epsilon_{\lambda\lambda} = \frac{1}{a}[u + \cot\theta\, v] \tag{43}$$

With this relation we can compute ϵ_a

$$S_a = S_1(h) + \cot\theta\; S_{2,\theta}(l) + S_{\theta\theta} \tag{44}$$

Remember, we expect that the finite sum behaves better with the first derivative included. If the sum behaves well without it, the areal strain will be simpler to compute

$$S_a = 2S_1(h) + S_{dd}(l) \tag{45}$$

The Kummer expression for areal strain is therefore

$$K_a = 2h_\infty K_1 + l_\infty K_3 \tag{46}$$

4 Fine tuning

The little deficit in neglecting the remainder between Love numbers at N and ∞ can be bridged by assuming

$$\gamma_n' = \gamma_n - c_\infty - \frac{c_N}{n+1} \qquad c_N = (N+1)(\gamma_N - \gamma_\infty) \tag{47}$$

As $n \to \infty$, and when $n = N$, $\gamma_n' = 0$. The $1/(n+1)$ behaviour is just a guess.

Doing Kummer transformations in the extended set, we obtain second order terms scaled by c_N. Our previous analytical terms are with the infinite Love numbers as before. But in the finite sums we must also subtract $c_N/(n+1)$. These numbers are BIG! (for Love numbers h for example, $c_N = 490$).

We need to compute a few new sums

$$K_5 \;:=\; \sum_{n=0}^{\infty} \frac{t^n}{n+1} P_n(\cos\theta) \tag{48}$$

$$K_{5,\theta} \;:=\; \sum_{n=1}^{\infty} \frac{t^n}{n+1} \frac{\partial P_n(\cos\theta)}{\partial\theta} \tag{49}$$

$$K_6 \;:=\; \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} P_n(\cos\theta) \tag{50}$$

$$K_{6,\theta} \;:=\; \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)} \frac{\partial P_n(\cos\theta)}{\partial\theta} \tag{51}$$

The results are given for $t = 1$. I got them with Mathematica.

$$K_5 = \log\left[1 + \frac{1}{\sin\theta/2}\right] \tag{52}$$

$$K_{5,\theta} = -\frac{\cot\theta/2}{2 + 2\sin\theta/2} \tag{53}$$

$$K_6 = 1 - 2\log\left[1 + \sin\theta/2\right] \tag{54}$$

$$K_{6,\theta} = -\frac{\cos\theta/2}{1 + \sin\theta/2} \tag{55}$$

If $t < 1$ the high-degree terms are suppressed, so that the long tail would not matter. You can check by doing the Kummer transformation with the Love numbers for $n = 10,000$ instead of ∞.