Stereographic Projection

hgs

1 The principle

The earth is considered to be a sphere of R=6,371 km radius. A point is chosen as the tangent point T of a plane touching the sphere. The plane is oriented to have the origo coincident with the tangent point, the j-axis coincident with the north direction and the i-axis at a direction 90°clockwise from the j-axis.

On the spherical earth we have the unit vectors $\hat{r}$,$\hat{\theta}$, $\hat{\lambda}$. Note that θ ("colatitude") is reckoned from the north pole, i.e. $\hat{\theta}$ points south.

The projection consists of two stages, (1) rotation of T to a place at (R,0,0); (2) projection to the plane.

1.1 Rotation

Let the point T on the sphere have east longitude λ_o and north latitude β_o. The following rotation matrix

$$\mathcal{R} = \begin{pmatrix} -\sin\beta_o\cos\lambda_o & -\sin\beta_o\sin\lambda_o & \cos\beta_o \\ -\sin\lambda_o & \cos\lambda_o & 0 \\ -\cos\beta_o\cos\lambda_o & -\cos\beta_o\sin\lambda_o & -\sin\beta_o \end{pmatrix}$$

will accomplish the referencing of an arbitrary point on the sphere in terms of latitude and longitude in the new system with its polar axis through T. The recipe for this operation is simply

$$\boldsymbol{z} = \mathcal{R}\boldsymbol{x}$$

1.2 Projection

The basic principle of the stereographic projection is shown in Figure 1. The positon vector of an arbitrary point on the sphere is represented by its spherical coordinates. In geocentric XYZ the position vector is

$$\boldsymbol{x} = (\sin\beta\cos\lambda, \sin\beta\sin\lambda, \cos\beta)^{\top}$$

After the rotation

$$\boldsymbol{z} = \mathcal{R}\boldsymbol{x}$$

we find the position e - east and n - north in the plane with respect to the origo by

$$\begin{aligned} e &= 2R\frac{z_2}{1-z_3} \\ n &= 2R\frac{z_1}{1-z_3} \end{aligned}$$

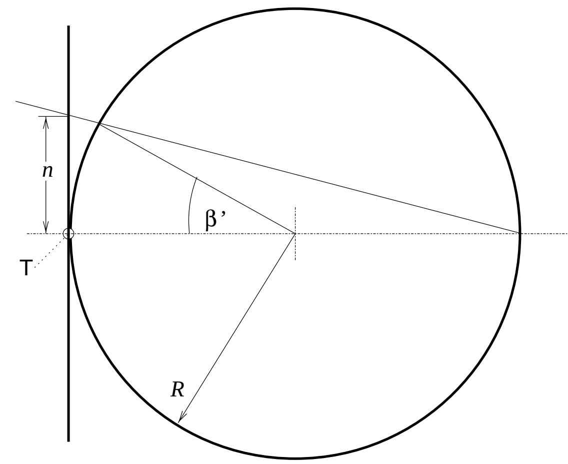

Figure 1: Principle of stereographic projection

1.3 Inverse projection

The inverse projection starts from e, n positions in the plane. Using

$$Z = 1 + \left(\frac{e}{2R}\right)^2 + \left(\frac{n}{2R}\right)^2$$

the rotated vector is

$$\boldsymbol{z} = \left(\frac{n}{Z}, \frac{e}{Z}, \frac{1}{2} - \frac{1}{Z}\right)^\top$$

from which one obtains

$$\boldsymbol{x} = \mathcal{R}^\top \boldsymbol{z}$$

and finally longitude and latitude from

$$\begin{aligned} \beta &= \tan^{-1} \frac{x_3}{\sqrt{x_1^2 + x_2^2}} \\ \lambda &= \tan^{-1} \frac{x_2}{x_1} \end{aligned}$$

For evaluation of the last equation the Fortran function `ATAN2(X(2),X(1))` must be used.