TTEQ - Ocean Tide and Storm Surge Equation Solver

Hans-Georg Scherneck[a]

[a]Chalmers - Department of Space and Geosciences and Onsala Space Observatory

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The storm surge equations are Laplace tidal equations in shallow water. The problem is explicitly time-dependent. I.e. a Fourier transformation into the frequency domain will not help. We have to go through a lengthy integration of the partial differential equations, especially if long-lasting forcings like a combination of tidal forces and air pressure and wind stress is to be considered.

It should be said already now that we will consider only the barotropic case. Thus you may object to the notion "storm surge" when we will be unable to resolve surface and bottom currents.

We will devote the major part of this document to the problem of wide-band tidal forcing, where different harmonic terms are allowed to interact owing the nonlinear terms occurring in the differential equations. We will develop them to one order beyond the linear approximation. This document will also cover the purpose of the preparatory stages, CREAM (Create A Model), otem92 (PREP-1) and otem16 (PREP2). (A name contest campaign is running; hot candidates for the winners are PETS for Prepare External Tide Sets, and CETI for Collect Explicitly Time- dependent Information.)

1 Momentum equations and continuity

The momentum equation in a rotating coordinate system states

$$\rho \left(\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} \right) = \vec{f} \tag{1}$$

and the force vector $\vec{f}$ consists of driving forces due to pressure and friction forces. The free boundary, the sea surface, will adjust such that the vertical flow counterbalances the divergence of the current. We integrate over a vertical column, ignoring the flow and density structure in the column to obtain

$$\partial_t \zeta = - \int_{-H}^{\zeta} \vec{\nabla}' \cdot \vec{u} \, dh \tag{2}$$

the continuity equation (supposing incompressibility of the sea water) where ζ is the sea surface elevation. The dashed gradient operator signifies that we only consider horizontal derivatives. This equation is the first instance of nonlinearity, since ζ appears in this expression as a factor on current. $H + \zeta$ will be disturbed especially in shallow water. And in a low-tide situation when $|\zeta| = H$ and$\zeta < 0$ the bottom is water free, so a sophisticated approach will be needed to cope with the clipping effect and the ensuing shoreline migration. We'll keep these complication out of our way for the time being.

Henceforth we will frequenly use the vertically integrated current, the transport vector

$$\vec{M} := \int_{-H}^{\zeta} \vec{u}\, dh$$

As to the forces, we start with pressure at an arbitrary depth h, $p = g\rho h$ and measure the elevation ζ from the equilibrium figure of the earth which includes the centrifugal potential

$$U = \frac{GM}{r} + \frac{1}{2}\Omega^2(r^2 - z^2)$$

The resting ocean will be aligned with the surface $U = const$. We can add here the tidal potential Ψ to U—if the ocean would instanteously adjust to an equipotential surface that includes the tide potential, no further mass flow would occur. The horizontal gradient of pressure due to the departure from the equipotential surface provides the current generating forces

$$\vec{f}_{\mathrm{tide}} = -g\rho\nabla\,(\zeta - \bar{\zeta}) \tag{3}$$

with the tidal equilibrium surface

$$\bar{\zeta} = \frac{1}{g}\Psi$$

and the forece on the water column is therefore

$$\vec{F}_{tide} = \int_{-H}^{\zeta} \vec{f}_{tide}\, dh = -g\rho(H + \zeta)\nabla\,(\zeta - \bar{\zeta})$$

Note that we obtain another shallow-water perturbation as ζ multiplies with the gradient of ζ.

And we can add atmospheric pressure forcing

$$\vec{F}_P = -(H + \zeta)\nabla\, p$$

And the friction forces, for which we may state three different laws

$$\vec{F}_f = -\eta\vec{u} - q|\vec{u}|\vec{u} + A_h\nabla^{\,2}\vec{u} \tag{4}$$

i.e. a linear law, a quadratic law, and a law that assumes turbulent friction to be proportional to second spatial derivatives of current. The latter is Boussinesq's approximation, and the factor A_h is the so-called horizontal austausch coefficient.

Equation (1) takes a Lagrangian perspective. Since we observe motion and elevation from a point fixed to the solid earth, the material derivative must be resolved into its partial time derivative and the advection term

$$\frac{D\vec{u}}{Dt} = \partial_t\vec{u} - \vec{u}\cdot\nabla\,\vec{u}$$

Vertically integrating, we collect everything above and arrive at the shallow-water equations

$$\partial_t\vec{M} = \frac{1}{H}\vec{M}\cdot\nabla\,\vec{M} - 2\vec{\Omega}\times\vec{M} - g(H + \zeta)\nabla\left(\frac{p}{g\rho} + \zeta - \bar{\zeta}\right) - \vec{F}_f \tag{5}$$

$$\partial_t\zeta = -\nabla\,\vec{M} \tag{6}$$

The Coriolis term can be handled in a simple way

$$2\vec{\Omega} \times \vec{M} = 2\Omega \sin(\beta)\ [\vec{M}]$$

where β is geographic latitude, and current

$$\vec{M} = \begin{pmatrix} M_x \\ M_y \end{pmatrix}$$

is rotated counterclockwise

$$[\vec{M}] = \begin{pmatrix} M_y \\ -M_x \end{pmatrix}$$

2 Boundary conditions

We distinguish between **passive boundaries** which are coastal features across which water flow is assumed inhibited, and **active boundaries** which connect the model basin to the world ocean or, more generally, where elevation or flow can be prescribed. In active boundaries we may drive the model with externally acquired tide information or we can step up elevation (or flow) and thus observe the basin's step response. Plulling the plug in the bath tub so to speak.

2.1 Passive boundaries

We demand the following simple condition at a coastal boundary; the on-shore flow must go to zero,

$$\vec{M} \cdot \hat{n} = 0 \tag{7}$$

where $\hat{n}$ is the unit vector perpendicular to the coastline.

The model may end at an interface where elevation and/or current can be prescribed. This feature is most meaningful in the case of period forcing, connecting the basins tides to world ocean tides that are supposed to be known well enough at the interface.

2.2 Active boundaries

Tidal information at active boundaries must be collected from external sources. This goes hand in hand with the computation of loading effects on the tide-raising potential due to the world ocean. The task is carried out in PREP-1 (otem91). When a spectrum of tide raising potential is designed, the range of partial wav is usually wider than what is covered by global ocean tide models. As far as possible, links are found via the astronomical argument numbers, and ocean tide parameters are selected from the available list by means of a nearest-neighbour-in-frequency principle and complex admittance.

$$Z_j = \zeta_k \frac{\Psi_j^*}{\Psi_{l(k)}^*}$$

where Z_j is a boundary value for tide constituent j, ζ_k a complex valued tide parameter of the global model suitably interpolated to the position of Z_j, and the Ψ's are the complex valued potential coefficients of the astronomical tide. The latter is found in the astronomical tide spectrum at $l(k)$. A link $j = j(k)$ is established such that the degrees and orders of the astronomical constituents match

$$n_j = n_{l(k)} \quad \text{and} \quad m_j = m_{l(k)}$$

and that the ocean tide frequency is not too far away.

$$\left|\omega_j - \omega_{l(k)}\right| = \min \forall\, k$$

Of course, one could interpolate the admittance spectrum $\zeta_k/\Psi^*_{l(k)}$. I shall not wipe second thoughts about this limitation off the table.

3 Self-attraction and self-loading (SAL)

Tide generation on an elastic earth means at first that the external potential is "filtered" with the elastic yield factor

$$\gamma_2 = 1 + k_2 - h_2$$

where k_2 and h_2 are the so-called body tide Love numbers; here we have restriced ourselves to the leading tide terms which come with spherical harmonic degree 2. Thus

$$\bar{\zeta} = \frac{\gamma_2}{g}\Psi_2(\beta, \lambda, t) + \text{higher order terms}$$

However, in additon to the primary tidal forces, the ocean masses both attract themselves through Newtonian attraction, and they deform the earth, so that there is an addition

$$\bar{\zeta}_{\text{SAL}} = \frac{G}{g}\int_{\mathcal{E}} \rho\zeta\mathcal{G}(\angle\vec{r}', \vec{r})\; ds'$$

where $\mathcal{G}(\angle\vec{r}', \vec{r})$ is the so-called loading Greens function

$$\mathcal{G}(\angle\vec{r}', \vec{r}) = \sum_{n=0}^{\infty}(1 + k'_n - h'_n)P_n(\cos(\angle\vec{r}', \vec{r}))$$

based on an infinite series of load Love numbers k'_n and h'_n.

A notorious difficulty with the SAL term is that it, strictly taken, must be evaluated at each time step. Owing to convolution, this is in the worst case a N^4 process. With Fast Fourier it can be done faster $[N\log(N)]^2$. If accuracy demands are advanced on the harmonic tide solutions, one possibility exists in taking a harmonic solution, convolving it with the Greens functions, to finally adding it to the tide-generating potential. This process can be iterated one or two times.

Convolving a harmonic solution for the basin madel is the task of LOAD (otem64). This stage needs a product from otem67, viz. the spatial (2D) spectrum of the Green's function.

Since I am a nerd in tidal loading (and since the whole idea of modelling was arose in the perspective of studying SAL) I preferred the plane grid approach in stereographic projection, with has the virtue of an equal-area mapping. That seemed plausible. At least it avoids some computational difficulties when grid cellss change their area with location.

For the Fourier method of load computation, a spectrum grid is produced of four times the size of the largest diameter of the basin in terms of grid nodes

$$S \geq 2\sqrt{M^2 + N^2}$$

where S preferably chosen as an integer power of 2. The Greens function values are filled only in the lower quadrant $((0,0) - -(S/2, S/2)$ in order to avoid circular correlation. The stereographic projection is employed to map the distance angle to grid units and vice versa. The Greens function is only dependent on distance but the area is cartesian, so we cannot utilize cylindrical symmetry. Utility otem67 prepares a 2D Fourier wavenumber transform of the kernel and stores it in a file. You will see frequent use of the term. tech. "Wave number" in the programs in context.

In otem64 the grid carraying the harmonic response of the basin is similarly exetended from $M \times N$ to $S \times S$ and then multipied node by node with the kernel spectrum. Then, by back-transform and crooping to $M \times N$ the SAL potential is obtained and stored in a file.

However, the parameterized SAL method might cope with modest demands on accuracy. otemt1 accepts a SAL parameter. A factor

$$f = (1 - p_{\mathrm{SAL}})$$

is multiplied with the elevations ζ in equation (3). A good value for the parameter has to be guessed. Since it appears in the gradient operator, the short-wavelength features are enhanced, so a recipe would be to check for a gain factor between the loading potential (converted into units of meter surface elevation) and elevation on the basis of gradients

$$\nabla \zeta_{\mathrm{SAL}} = p_{\mathrm{SAL}} \nabla \zeta + \epsilon$$

in a least-square sense. Use the output of otem64.

We have been referred to notions of **projections** and **grids**. These will be detailed in the next sections.

4 Stereographic projection

The earth is considered to be a sphere of R=6,371 km radius. A point is chosen as the tangent point T of a plane touching the sphere. The plane is oriented to have the origo coincident with the tangent point, the j-axis coincident with the north direction and the i-axis at a direction 90°clockwise from the j-axis.

On the spherical earth we have the unit vectors $\hat{r}, \hat{\theta}, \hat{\lambda}$. Note that θ ("colatitude") is reckoned from the north pole, i.e. $\hat{\theta}$ points south.

The projection consists of two stages, (1) rotation of T to a place at (R,0,0); (2) projection to the plane.

4.1 Rotation

Let the point T on the sphere have east longitude λ_o and north latitude β_o. The following rotation matrix

$$\mathcal{R} = \begin{pmatrix} -\sin\beta_o \cos\lambda_o & -\sin\beta_o \sin\lambda_o & \cos\beta_o \\ -\sin\lambda_o & \cos\lambda_o & 0 \\ -\cos\beta_o \cos\lambda_o & -\cos\beta_o \sin\lambda_o & -\sin\beta_o \end{pmatrix}$$

will accomplish the referencing of an arbitrary point on the sphere in terms of latitude and longitude in the new system with its polar axis through T. The recipe for this operation is simply

$$\boldsymbol{z} = \mathcal{R}\boldsymbol{x}$$

4.2 Projection

The basic principle of the stereographic projection is shown in Figure 1. The positon vector of an arbitrary point on the sphere is represented by its spherical coordinates. In geocentric XYZ the position vector is

$$\boldsymbol{x} = (\sin\beta\cos\lambda, \sin\beta\sin\lambda, \cos\beta)^\top$$

After the rotation

$$\boldsymbol{z} = \mathcal{R}\boldsymbol{x}$$

we find the position e - east and n - north in the plane with respect to the origo by

$$\begin{aligned} e &= 2R\frac{z_2}{1-z_3} \\ n &= 2R\frac{z_1}{1-z_3} \end{aligned}$$

4.3 Inverse projection

The inverse projection starts from e, n positions in the plane. Using

$$Z = 1 + \left(\frac{e}{2R}\right)^2 + \left(\frac{n}{2R}\right)^2$$

the rotated vector is

$$\boldsymbol{z} = \left(\frac{n}{Z}, \frac{e}{Z}, \frac{1}{2} - \frac{1}{Z}\right)^\top$$

from which one obtains

$$\boldsymbol{x} = \mathcal{R}^\top\boldsymbol{z}$$

and finally longitude and latitude from

$$\begin{aligned} \beta &= \tan^{-1}\frac{x_3}{\sqrt{x_1^2 + x_2^2}} \\ \lambda &= \tan^{-1}\frac{x_2}{x_1} \end{aligned}$$

For evaluation of the last equation the Fortran function `ATAN2(X(2),X(1))` must be used.

5 Discretisation

Small regions can be modelled with a regular plane grid. A diagonal staggering is an efficient method to interleave the current and elevation grids, see Fig. 2 As a consequence, the basic grid directions are turned counterclockwise by 45°with respect to an east-north orientation. Boundaries are still aligned with east and

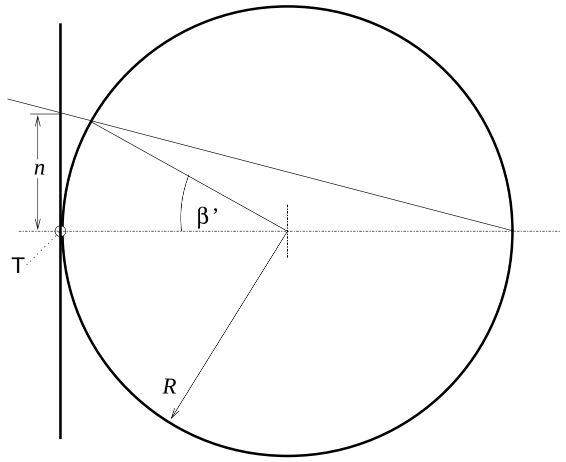

Figure 1: Principle of stereographic projection

north, so that a straight stretch of e.g. north-south running coastline demandes that $M_x + M_y = 0$, and equavalently $M_x - M_y = 0$ at an east-west running coastline. At capes the one of the diagonal components that heads at the corner is set to zero. And in the corner point of a bay, both components are set to zero.

Open boundaries are preferentially straight connections of grid points between a pair land points A and B. Many such pairs can be set up. However, more complicated geometries are possible, just a little more tedious to formulate.

6 Data

In this section all kinds of parameters will be detailed, for several purposes and reasons. First, to document what TTEQ is doing, what assumtions go in. Second, to help others to construct similar procedures so that results can be reproduced and/or critically tested. Third, to provide a catalogue of data sources meeting more general curiosity.

6.1 Bathymetry

TTEQ's preparation stage CREAM can process TOPO05 and TerrainBase, two 5'×5' topography-bathymetry data bases, or ETOPO1, a 1'×1' data base.

6.2 Global Ocean Tides

The recent versions of the OTEQ/TTEQ package accept netCDF files. Thus the model can be driven with a whole range of sources, from FES2004 (1/8°×1/8°) or Schwiderski (1°×1°). Mixing of driving sources, however, is not yet tested and probably not yet possible.

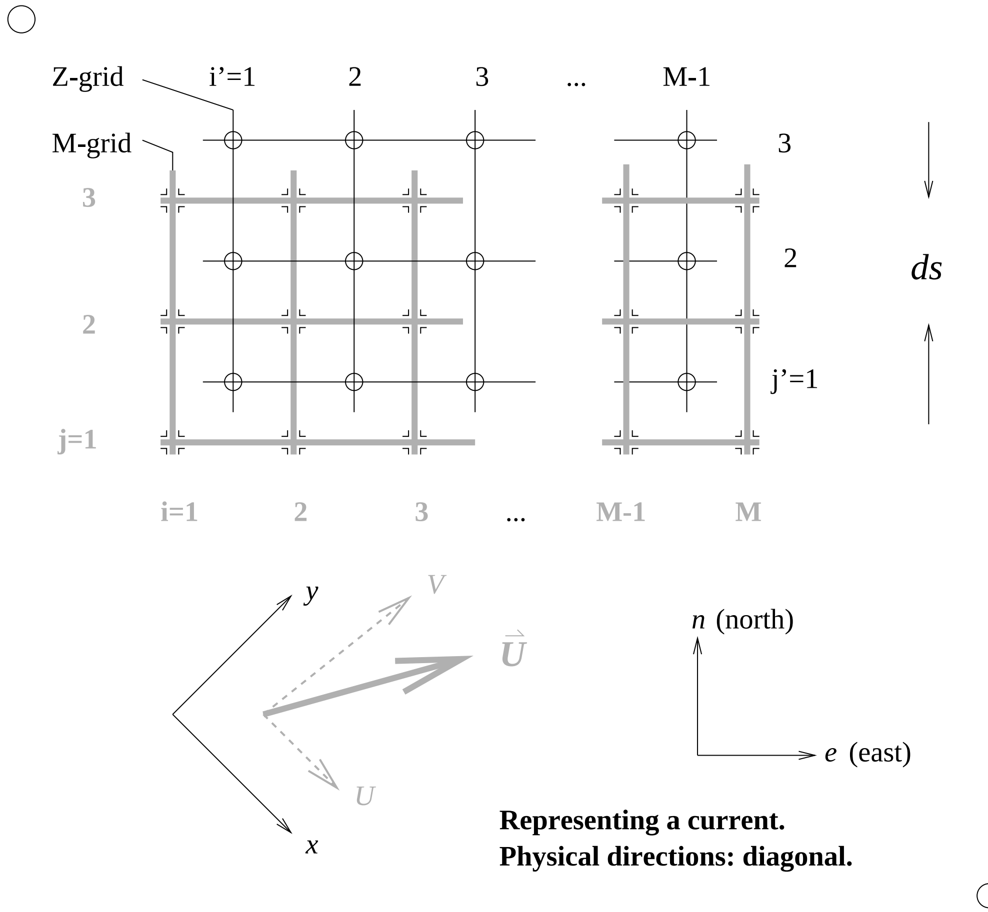

Figure 2: Staggered grids for first-order coupled PDE's. On the Z-grid the sea level, surface pressure and tidel equilibrium elevations are given, and on the M-grid the transport vectors are given. The gradients of the advection term are calculated from near-neighbour differences in upwind direction from the actual node.

6.3 Tide generating potential

The current version uses a potential development by Tamura. The data file is slightly specific for the epoch. Its name is `Ttide/etgtab/ATS.dat`. Leap-second information must be specified at the otem16 stage, more precisely in the namelist, if considered critical. The computation of the tide potential is carried out under OTEQ, so that the `ATS.dat` file is needed then. The routines that is called are `OTEP` (otes913.f) and `OTEPRC` (otes911.f).

A fresh file `ATS.dat` can be produced with `ttimm ttimm.ins '>BT24>'` (in directory `Ttide`).

The relevant subroutines are `otes16*.f` during preparation and `otes17*.f` during time stepping. Subroutine `ETDPRP` (otes16.f) under Main (otem16.f) calls `ASTRO` (thtide.f) for amplitudes, frequencies and phases. `ETDCMP` (otes17.f) under TTEQ (Main e.g. otemt*.f) computes the argument of date and the geodetic coefficients.

Table 1: Constants adopted in the OTEQ/TTEQ package

Symbol	Meaning	Value	Units
Ω	Sidereal earth rotation	1.002738	cyc/day
f_{O1}	O_1 tide frequency	0.929536	cyc/day
f_{NDFW}	NDFW resonance frequency	1.00507	cyc/day
S_γ	NDFW resonance strength	-0.001230	-
$\gamma_{2,1}(f_{\mathrm{O1}})$	elastic earth factor	0.695	-
$\gamma_{20},\ \gamma_{2,2}$	elastic earth factor	0.695	-
γ_{3*}	elastic earth factor	0.805	-
γ_{4*}	elastic earth factor	0.869	-
D	Doodson's constant	2.6277	m^2/s^2
g	Surface gravity	9.81	m/s^2
a or R	Earth radius	6,371	km
G	Gravity constant	$6.671\times10-11$	$m^3/kg/s^2$

As a side remark, in TTEQ I had been using the tide potential of Büllesfeld, an augmentation to Cartwright and Tayler's table, for a long time. In February 2011 I have adapted the package to the Tamura potential. I found major differences in the coefficients for e.g. M_3, i.e. at degrees greater than two. Yet, the factors and Legendre polynomials applied to the coefficients do not appear to change between the subroutines `astros.f` and `thtide.f` (Büllesfeld). The numbers are fairly well in line with Tamuras publication [Tamura, 1978], so I might have made big mistakes in tide analysis before (I have not computed M_3 tides though.

The elastic earth factor γ_n, e.g. for a semi-diurnal tide of degree n=2

$$\gamma_2 = 1 + k_2 - h_2$$

is computed from a spectrum parametrisation in subroutine (`BTAMPG,` oteu911.f). It takes the NDFW into account. Actually, it is a γ_{nm} that is computed owing to the work of Wahr on the Nearly Diurnal Free Wobble (NDFW)

$$\gamma_{21} = \gamma_{21}(f) = \gamma_{21}(f_{\mathrm{O1}}) \left(1 + S_\gamma \frac{f - f_{\mathrm{O1}}}{f - f_{\mathrm{NDFW}}}\right)$$

with S_γ resonance strength and f_{NDFW} resonance frequency.

Loading effects are calculated on the basis of the Greens functions of Farrell. Since TTEQ considers loading effects within a narrow range of distances, the Greens functions are sometimes interpolated and kept in tables to save computation time for better purposes. Some tables (which?) expand over distance not based on the usual logarithmic but rather on a hyperbolic law. Since loading Greens functions regularly possess this notorious singularity at zero distance, it was chosen to table values as factors on the asymptote. In the case of the tide-generating potential, the asymptotic function is $1/(2\sin\theta/2)$. (The hyperbola was once drawn out of the sleeve. It does not seem to matter, and today I would use logarithms throughout.) The asymptotic function is always computed at the precise distance; the table is looked up by the nearest-neighbour principle.

The computation of global loading effects upon the target area is a quite heavy task. The subroutine employed for this work, `AGOTEP` (otes92.f), uses some speeding-up tricks (sincere tricks, not cheating), namely Fourier transforms; thus, convolution of a pair of latitude rings can be reduced to multiplication in the spectral domain. Here-in, circular convolution is fortuitiously exploited. The code puts an effort to keep the number of recomputations of the Greens function spectrum low, utilising hemispherical symmetries.

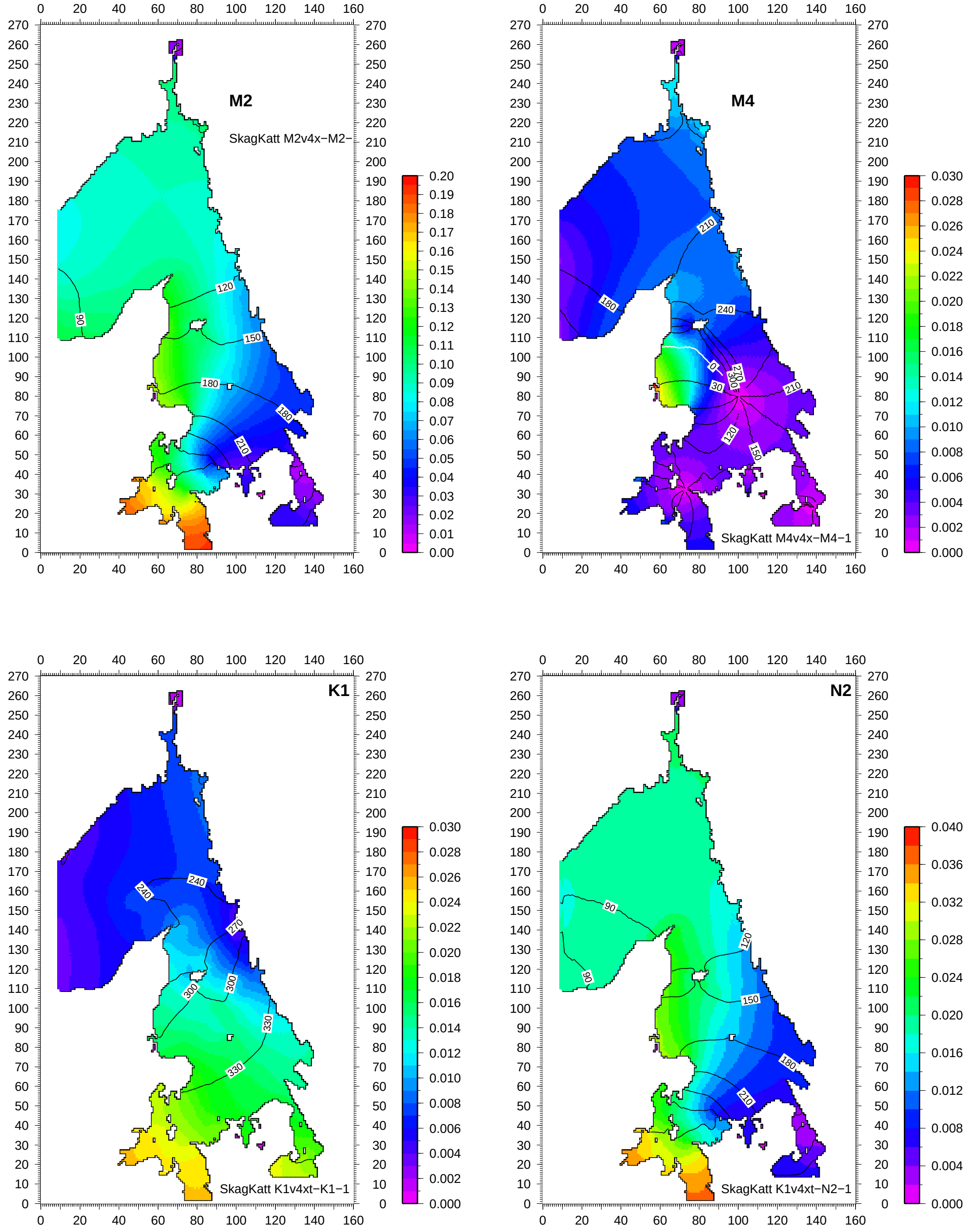

Figure 3: Top row: Solutions for M_2 and M_4 due to exitation with FES2004 on the western boundary in Skagerak. Bottom row: K_1 and N_2, excited with TPXO.7.2

7 Examples

We show some examples for a Skagerak-Kattegat model in a few figures 3. Very incomplete collection (depth and potential should be plotted too).

8 Final remarks

Detailed documentation of ”How-to” character is unfortunately a bit scattered at `http://froste.oso.chalmers.se/hgs/OTEQ` with some overview material in the top of the list. The `*.doc` files are not, as intuition might suggest, Word-documents but rather text files that should be looked at in a plain MS-DOS window (wide screen) or with QuickViewPlus (view in MS-DOC mode). Then you will see line drawing characters illustrating the passive boundaries in figures and examples. There is also much if not most documentation for OTEQ routines, i.e. programs that aim at computation of a single harmonic tide.

This document is to be extended to instruct for the use of atmospheric pressure fields in excitation. And there is a wind-forced version, SSEQ (`otem12w.f`), SSEQ for storm-surge equations.

Still missing here are all kinds of explicit references. I should say in leaving that I have learnt masses from the Diploma thesis of Carsten Wübber, Inst. Meereskunde Kiel), on the Seiches of the Baltic Sea [Wübber and Krauss,1979]; I owe hom the staggered grids. I also gained from a paper of Roger A. Flather’s (Bidston, U.K.). The textbooks by W. Krauss on theoretical physical oceanography and L. Collatz on differential equations. For a period of time I had “austausch” with Wilfried Zahel at Meereskunde in Hamburg, who gave me many impulses and loads worth of suggestions on friction and differential equation solving. I am still not sure whether I have implemented the austausch coefficient and eddy friction in a correct/sensible fashion. If you have suggestions, please contact me at hgs “at” chalmers.se

> The program package has its own history, a Malstrom that met both Scylla and Charybdis. Back in the late 1980’ies I coded the first lines in Uppsala on an IBM 370 mainframe churning away at my and hundred other people’s code in the sacred interior of UDAC, the university’s data centre. The program ran under the GUTS (Gothenburg University Timesharing System). This was tedious, except that the system endorsed the linedraw characters. I could use DISSPLA for plotting, but then I had to commute to the data centre to collect the drawings. For quicker access I had to download ascii files and draft them on a plotter connected to a Luxor ABC-80 system for which I had to code the plotting interface. In a transition period I might have been able to get portable graphics files out of the DISSPLA post processor, I have forgotten the details; portable meaning to the equipment at the Section for Geodesy at Hällby by telephone modem (we once celebrated the arrival of a true full-duplex 1200 baud system!). Later on I bought an IBM PC on which I could run Surfer/Grapher, and instruct a Hitachi flatbed plotter; also for this machine I had to code the interface as it wasn’t on Surfer’s device list. Still later I bought a 386 PC equipped with extended memory, and in those days it was modern to use Quarterdeck Desqview and QEMM, their memory manager. I brought all the code to the PC, bought the Microway Fortran compiler (with the grex graphics library), PharLap linker tools, and continued with Grapher/Surfer for plots on the Hitachi. Now I had beautiful screen graphics but no device that could capture and print them except my camera on a tripod and the local photo shop that developped the films.
>
> In 1993 I moved to Chalmers. Ocean tides were not high on the list of items to work on. But eventually, after abandoning the 386 that I had brought with me, I got my progams running on a short series of Unix machines, first on HP-Apollo RISCs, then on the Linux boxes that came to replace mainframes and clusters. Porting the package to Unix meant a great deal of reprogramming. My 386-PC graphics had to be adapted to the new platforms anyway, so I coded up a mock-up of Microway’s grex that now would use PGPLOT for graphics and Curses for text screen interactions. I you want to install TTEQ you’ll have to go through the Scylla or Carybdis to install an old version of PGPLOT (since I had to recode the Xwindows driver to accept more keys and mouse actions that Jim Parsons had envisioned in the mid 90-ies. Now, PGPLOT can do so much more, and my windows are always limited to 640×480 pixels. Recoding the 10000+ lines to skip grex would be a year worth of hacking. No, thanks.

I should say finally, that TTEQ and the package are tasty bits for an Absoft Fortran 77 compiler. Yes, you have read correctly.